

Differential rotation of nonlinear r-modes

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Differential rotation of r-modes is investigated within the nonlinear theory up to second order in the mode amplitude in the case of a slowly-rotating, Newtonian, barotropic, perfect-fluid star. We find a nonlinear extension of the linear r-mode, which represents differential rotation that produces large scale drifts of fluid elements along stellar latitudes. This solution includes a piece induced by first-order quantities and another one which is a pure second-order effect. Since the latter is stratified on cylinders, it cannot cancel differential rotation induced by first-order quantities, which is not stratified on cylinders. It is shown that, unlikely the situation in the linearized theory, r-modes do not preserve vorticity of fluid elements at second-order. It is also shown that the physical angular momentum and energy of the perturbation are, in general, different from the corresponding canonical quantities.

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I. INTRODUCTION

R-modes in Newtonian gravity were first studied more than twenty years ago [1, 2, 3]. In recent years, it was discovered numerically [4], and soon afterwards confirmed analytically [5], that these modes, analogous to Rossby waves in the Earth's atmosphere and oceans, are driven unstable by gravitational radiation reaction in perfect fluid stars with arbitrary small rotation.

The interest in the astrophysical implications of r-modes increased dramatically when it was shown [6, 7] that in a newly born, hot, rapidly-rotating neutron star the radiation reaction force dominates bulk and shear viscosity for enough time to allow most of the star's angular momentum to be radiated away as gravitational waves. As a result, the neutron star spins down to just a small fraction of its initial angular velocity, thus providing a possible explanation for the relatively small spin rates of young pulsars in supernova remnants. For typical equations of state of a neutron star it was estimated [6, 7] that the r-mode instability spins down a young neutron star to a period of about 10–20 ms. This is comparable to the inferred initial periods P_0 of the fastest pulsars associated with supernova remnants, such as the Crab pulsar PSR B0531+21 ($P_0 = 19$ ms), or PSR J0537-6910 ($P_0 \ll 16$ ms for braking index $n = 3$ [8]). The detectability of gravitational waves from such sources have been analyzed within a simple phenomenological model for the evolution of the r-mode instability, with the conclusion that gravitational waves emitted from an young, rapidly rotating neutron star in the Virgo cluster could be detected by enhanced versions of laser interferometer detectors [9]. However, recent results on the nonlinear saturation of the r-mode energy indicate that enhanced LIGO detectors could detect gravitational waves from these sources only up to a distance of about 200 kpc [10].

Another interesting astrophysical implication of r-modes is related to the fact that gravitational-wave emission due to mode instability could balance the spin-up torque due to accretion of neutron stars in low-mass X-ray binaries [11, 12], thus limiting the maximum angular velocity of these stars to values consistent with observations (recent results indicate that the fast end of the spin distribution could be around 600 Hz, well below the theoretical maximum [13]). If the viscosity of the neutron star is a decreasing function of temperature, then the star undergoes a cyclical evolution of spin-up due to accretion and spin-down due to gravitational radiation emission [14]. Assuming that the saturation amplitude of the r-mode is of order unity, it was estimated [14] that the time spent by the star in the spin-down phase is about one year, just a small fraction of the full duration of the cycle (several million years), thus making it unlikely that any such source in our Galaxy is presently emitting gravitational waves. However, recent results indicate that the saturation amplitude of the r-mode could be much smaller than unity, implying that the duration of the spin-down phase could be a much larger fraction of the cycle [15]; in that case, gravitational radiation from such sources in our galaxy could be detected by enhanced LIGO detectors [10].

While initial research on r-modes was carried out within the linearized theory, a deeper understanding of these modes and its astrophysical implications requires considering the nonlinear theory.

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An important nonlinear effect, that has been investigated recently by several authors, is differential rotation induced by r-modes. This differential rotation could wind up a magnetic field in an old, accreting neutron star in an X-ray binary, leading to a gamma-ray burst [16]. Differential rotation could also interact with the magnetic field of a newly born, rapidly rotating neutron star, limiting the growth of the r-mode instability or, for strong magnetic fields, even preventing it from developing [17, 18, 19].

An analytical expression for differential rotation induced by r-modes was first derived using the linearized fluid equations by expanding the velocity of a fluid element located at a certain point in powers of the mode amplitude, averaging over a gyration, and retaining only the lowest-order nonvanishing term [17]. Since this procedure is not equivalent to solving the nonlinear fluid equations, the analytical expression thus obtained is just an approximate one. Differential rotation was also reported in a useful toy model of a thin spherical shell of a rotating incompressible fluid [20]. However, in this case, differential rotation is of a different nature than the one referred to above; it is driven by radiation reaction. The existence of differential rotation related to r-modes was confirmed by numerical simulations [21, 22].

In this paper, differential rotation induced by r-modes is investigated within the nonlinear theory up to second order in the mode amplitude α in the case of a Newtonian, barotropic, perfect-fluid star rotating with constant angular velocity Ω . Second-order quantities are obtained by expanding simultaneously in powers of α and Ω . In order to consistently neglect higher-order terms arising from the expansion in Ω , it is assumed that $\alpha \gg (\Omega/\Omega_K)^2$, where Ω_K is the angular velocity at which the star starts shedding mass through the equator. The investigation of r-modes up to second order in the angular velocity Ω , neglecting higher-order terms arising from the expansion in α , was carried out in Ref. [7] (using Saio's formalism [3]) and in Ref. [23] (using the two-potential formalism [24]). In order to neglect the deformation of the star due to the centrifugal force, our analysis is carried out in a slow-rotation approximation, $\Omega \ll \Omega_K$.

In section II we review some important results obtained in the linear theory of r-modes. In section III, a nonlinear extension of the linear r-mode perturbation is derived up to second order in the mode amplitude α . This solution describes differential rotation that produces large scale drifts of fluid elements along stellar latitudes. The Lagrangian theory of nonrelativistic fluids [25] is used in this section to apply the boundary condition at the surface of the star and to show that, at second-order, r-modes do not preserve vorticity of fluid elements. The physical angular momentum and energy of the second-order r-mode solution are analyzed in section IV. In particular, it is shown that, in general, physical and canonical quantities are different. Finally, section V is devoted to discussion and conclusions.

II. R-MODES IN THE LINEARIZED THEORY

The linearized fluid equations for an uniformly rotating, Newtonian, barotropic, perfect-fluid star in an inertial frame are

$$\partial_t \delta^{(1)} v_i + \delta^{(1)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(1)} v_i = -\nabla_i \left(\frac{\delta^{(1)} p}{\rho} + \delta^{(1)} \Phi \right), \quad (1)$$

$$\partial_t \delta^{(1)} \rho + v^i \nabla_i \delta^{(1)} \rho + \nabla_i (\rho \delta^{(1)} v^i) = 0, \quad (2)$$

$$\nabla^i \nabla_i \delta^{(1)} \Phi = 4\pi G \delta^{(1)} \rho, \quad (3)$$

where $v^i = \Omega \delta_\phi^i$ and ρ are, respectively, the fluid velocity and the mass density of the unperturbed star, ρ is related to the pressure p by an equation of state of the form $p = p(\rho)$, and the quantities $\delta^{(1)} v^i$, $\delta^{(1)} p$, $\delta^{(1)} \rho$, and $\delta^{(1)} \Phi$ are, respectively, the first-order Eulerian change in velocity, pressure, density, and gravitational potential.

At lowest order in Ω the linearized Euler equation (1) yields, in spherical coordinates (r, θ, ϕ) , the r-mode solution

$$\delta^{(1)} v^r = 0, \quad (4a)$$

$$\delta^{(1)} v^\theta = \alpha \Omega C_l l (r/R)^{l-1} \sin^{l-1} \theta \sin(l\phi + \omega t), \quad (4b)$$

$$\delta^{(1)} v^\phi = \alpha \Omega C_l l (r/R)^{l-1} \sin^{l-2} \theta \cos \theta \cos(l\phi + \omega t), \quad (4c)$$

with $\delta^{(1)} U \equiv \delta^{(1)} p/\rho + \delta^{(1)} \Phi$ given by

$$\delta^{(1)} U = 2\alpha \Omega^2 \frac{C_l l}{l+1} R^2 \left(\frac{r}{R} \right)^{l+1} \sin^l \theta \cos \theta \cos(l\phi + \omega t), \quad (5)$$

where $\omega = -\Omega + 2\Omega/(l+1)$ is the mode angular frequency in the inertial frame, R is the radius of the unperturbed star and $C_l = (2l-1)!! \sqrt{(2l+1)/[2\pi(2l)!!(l+1)]}$. We will only consider modes for which $l \geq 2$.

The above solution satisfies the linearized continuity equation (2), which at lowest order in Ω is simply $\nabla_i (\rho \delta^{(1)} v^i) = 0$.

The linearized Euler and continuity equations do not contain information on how to split $\delta^{(1)}U$ into the first-order changes in pressure $\delta^{(1)}p$ and gravitational potential $\delta^{(1)}\Phi$. This has to be done by using the perturbed Poisson equation (3) and the boundary conditions for the gravitational potential at the surface of the star and at infinity. Assuming for $\delta^{(1)}\Phi$ the same angular dependence as $\delta^{(1)}U$, its (dimensionless) radial part $f(r)$ must be a solution of the equation [23]

$$\frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{df(r)}{dr} + \left(4\pi G \rho \frac{d\rho}{dp} - \frac{(l+1)(l+2)}{r^2} \right) f(r) = 4\pi G \rho \frac{d\rho}{dp} \left(\frac{r}{R} \right)^{l+1}, \quad (6)$$

with the conditions that $\delta^{(1)}\Phi$ be continuous at the surface of the star, have a continuous first derivative there and tend to zero as $r \rightarrow \infty$.

The r-mode solution given by eqs. (4) and (5) also satisfies the zero-boundary condition for the pressure p at the surface of the star, i.e., the first-order Lagrangian change in pressure [28] vanishes at the surface of the star, $\Delta_\xi^{(1)}p = 0$.

The Lagrangian change in pressure is, at first order, given by

$$\Delta_\xi^{(1)}p = -\gamma p \nabla_i \xi^{(1)i}, \quad (7)$$

where $\xi^{(1)i}$ are the contravariant components of the first-order Lagrangian displacement vector and $\gamma = (\rho/p)(\partial p/\partial \rho)$. These components are the solution of the equations [25]

$$\delta^{(1)}v^i = \partial_t \xi^{(1)i} + v^k \nabla_k \xi^{(1)i} - \xi^{(1)m} \nabla_m v^i = (\partial_t + \Omega \partial_\phi) \xi^{(1)i}, \quad (8)$$

$$\delta^{(1)}\rho = -\nabla_i (\rho \xi^{(1)i}), \quad (9)$$

where $\partial_t + \Omega \partial_\phi$ is the time derivative in the rotating frame, $\partial_{t'}$. At lowest order in Ω these equations yield for $\xi^{(1)i}$ the following solution:

$$\xi^{(1)r} = 0, \quad (10a)$$

$$\xi^{(1)\theta} = -\frac{1}{2} \alpha C_l l(l+1) \left(\frac{r}{R} \right)^{l-1} \sin^{l-1} \theta \cos(l\phi + \omega t), \quad (10b)$$

$$\xi^{(1)\phi} = \frac{1}{2} \alpha C_l l(l+1) \left(\frac{r}{R} \right)^{l-1} \sin^{l-2} \theta \cos \theta \sin(l\phi + \omega t). \quad (10c)$$

Using the above $\xi^{(1)i}$ it is now straightforward to verify that the Lagrangian change in pressure $\Delta_\xi^{(1)}p$ vanishes everywhere in the star, including the surface.

The Lagrangian displacement $\xi^{(1)i}$, given by eqs. (10), is canonical [25]; it satisfies the conditions $\epsilon^{ijk} \nabla_j \Delta_\xi^{(1)} v_k = 0$, where the first-order Lagrangian change in velocity is given by

$$\Delta_\xi^{(1)} v_i = \partial_t \xi_i^{(1)} + v^k \left(\nabla_i \xi_k^{(1)} + \nabla_k \xi_i^{(1)} \right). \quad (11)$$

For the canonical Lagrangian displacement (10) the canonical energy

$$E_c = \frac{1}{2} \int \left[\rho \partial_t \xi^{(1)i} \partial_t \xi_i^{(1)} - \rho v^j \nabla_j \xi^{(1)i} v^k \nabla_k \xi_i^{(1)} + \gamma p (\nabla_i \xi^{(1)i})^2 + 2 \xi^{(1)i} \nabla_i p \nabla_j \xi^{(1)j} + \xi^{(1)i} \xi^{(1)j} (\nabla_i \nabla_j p + \rho \nabla_i \nabla_j \Phi) \right] dV + \mathcal{O}(\Omega^4), \quad (12)$$

is negative for any $l \geq 2$ and for arbitrarily slow rotation [5], implying r-modes instability to gravitational radiation [26]. Let us emphasize here that the use of this stability criterion does not require knowledge of second-order Lagrangian displacements, since the canonical energy is quadratic in first-order quantities.

Using the relations $\Delta_\xi^{(1)} \epsilon^{ijk} = -\epsilon^{ijk} \nabla_m \xi^{(1)m} = 0$ and $\Delta_\xi^{(1)} \nabla_j v_k = \nabla_j \Delta_\xi^{(1)} v_k - v^m \nabla_j \nabla_k \xi_m^{(1)}$, it immediately follows that the canonical condition $\epsilon^{ijk} \nabla_j \Delta_\xi^{(1)} v_k = 0$ is equivalent to the statement that the first-order Lagrangian change in vorticity is zero, $\Delta_\xi^{(1)} (\epsilon^{ijk} \nabla_j v_k) = 0$. Thus, canonical displacements are precisely those that preserve vorticity. If the vorticity of a r-mode perturbation is initially small, then as the perturbation grows under gravitational radiation reaction the vorticity will not grow.

III. THE SECOND-ORDER R-MODE SOLUTION

At second order, the perturbed Euler, continuity, and Poisson equations in an inertial frame are given, respectively, by

$$\partial_t \delta^{(2)} v_i + \delta^{(2)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(2)} v_i + \delta^{(1)} v^k \nabla_k \delta^{(1)} v_i = -\nabla_i \delta^{(2)} U + \frac{\delta^{(1)} \rho}{\rho} \nabla_i \left(\frac{\delta^{(1)} p}{\rho} \right), \quad (13)$$

$$\partial_t \delta^{(2)} \rho + v^i \nabla_i \delta^{(2)} \rho + \nabla_i (\rho \delta^{(2)} v^i) + \nabla_i (\delta^{(1)} \rho \delta^{(1)} v^i) = 0, \quad (14)$$

$$\nabla^i \nabla_i \delta^{(2)} \Phi = 4\pi G \delta^{(2)} \rho, \quad (15)$$

where $\delta^{(2)} U \equiv \delta^{(2)} p / \rho + \delta^{(2)} \Phi$ and $\delta^{(2)} q$ denotes the second-order Eulerian change in a quantity q .

Let us now assume that $\alpha \gg (\Omega / \Omega_K)^2$. Then, terms in $\delta^{(2)} v^i$ proportional to $\alpha \Omega^3$ (arising in an expansion in powers of the angular velocity of the star) can be neglected when compared with terms proportional to $\alpha^2 \Omega$ (arising in an expansion in powers of the mode amplitude). For the same reason, we neglect in $\delta^{(2)} U$ and $\delta^{(2)} \rho$ terms proportional to $\alpha \Omega^4$. Since the second term on the right-hand side of eq. (13) is of order $\alpha^2 \Omega^4$, equation (13) reduces, at lowest order in Ω , to

$$\partial_t \delta^{(2)} v_i + \delta^{(2)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(2)} v_i + \nabla_i \delta^{(2)} U = -\delta^{(1)} v^k \nabla_k \delta^{(1)} v_i, \quad (16)$$

where $\delta^{(1)} v^i$ is of order $\alpha \Omega$, $\delta^{(2)} v^i$ is of order $\alpha^2 \Omega$ and $\delta^{(2)} U$ is of order $\alpha^2 \Omega^2$. In this equation second-order quantities depend on the first-order ones only through the term quadratic on $\delta^{(1)} v^i$. Using eqs. (4) to compute this term, the above equation reads, in spherical coordinates (r, θ, ϕ) ,

$$\begin{aligned} (\partial_t + \Omega \partial_\phi) \delta^{(2)} v^r - 2\Omega r \sin^2 \theta \delta^{(2)} v^\phi + \partial_r \delta^{(2)} U = \\ -\frac{1}{2} \alpha^2 \Omega^2 C_l^2 l^2 R \left(\frac{r}{R} \right)^{2l-1} \sin^{2l-2} \theta \{ \sin^2 \theta - 2 + \sin^2 \theta \cos[2(l\phi + \omega t)] \}, \end{aligned} \quad (17a)$$

$$\begin{aligned} (\partial_t + \Omega \partial_\phi) \delta^{(2)} v^\theta - 2\Omega \sin \theta \cos \theta \delta^{(2)} v^\phi + \frac{1}{r^2} \partial_\theta \delta^{(2)} U = \\ -\frac{1}{2} \alpha^2 \Omega^2 C_l^2 l^2 \left(\frac{r}{R} \right)^{2l-2} \sin^{2l-3} \theta \cos \theta \{ \sin^2 \theta + 2l - 2 + \sin^2 \theta \cos[2(l\phi + \omega t)] \}, \end{aligned} \quad (17b)$$

$$\begin{aligned} (\partial_t + \Omega \partial_\phi) \delta^{(2)} v^\phi + 2\Omega \frac{\delta^{(2)} v^r}{r} + 2\Omega \frac{\cos \theta}{\sin \theta} \delta^{(2)} v^\theta + \frac{1}{r^2 \sin^2 \theta} \partial_\phi \delta^{(2)} U = \\ + \frac{1}{2} \alpha^2 \Omega^2 C_l^2 l^2 \left(\frac{r}{R} \right)^{2l-2} \sin^{2l-2} \theta \sin[2(l\phi + \omega t)]. \end{aligned} \quad (17c)$$

The right-hand side of the previous equations contains a piece that does not depend on t and ϕ and a double “frequency” $2(l\phi + \omega t)$ piece. The former induces an axisymmetric time-independent second-order solution corresponding to differential rotation, while the latter induces a second-order solution corresponding to an oscillating response at a “frequency” twice that of the r-mode. In this article we will be concerned exclusively with the solution corresponding to differential rotation of r-modes.

The axisymmetric time-independent right-hand side of system (17) induces the following second order solution:

$$\delta^{(2)} v^r = 0, \quad (18a)$$

$$\delta^{(2)} v^\theta = 0, \quad (18b)$$

$$\delta^{(2)} v^\phi = \frac{1}{2} \alpha^2 \Omega C_l^2 l^2 (l^2 - 1) \left(\frac{r}{R} \right)^{2l-2} \sin^{2l-4} \theta, \quad (18c)$$

and

$$\delta^{(2)} U = -\frac{1}{4} \alpha^2 \Omega^2 C_l^2 l R^2 \left(\frac{r}{R} \right)^{2l} \sin^{2l-2} \theta [\sin^2 \theta - 2l^2]. \quad (19)$$

Indeed, assuming axisymmetry and time-independence system (17) decouples into two independent systems, one determining $\delta^{(2)} v^\phi$ and $\delta^{(2)} U$, the other relating $\delta^{(2)} v^r$ and $\delta^{(2)} v^\theta$. Eliminating $\delta^{(2)} U$ from eqs. (17a) and (17b), one obtains an equation that determines $\delta^{(2)} v^\phi$, namely

$$\Omega \sin \theta \cos \theta \partial_r \left(r^2 \delta^{(2)} v^\phi \right) - \Omega r \partial_\theta \left(\sin^2 \theta \delta^{(2)} v^\phi \right) = \alpha^2 \Omega^2 C_l^2 l^2 (l^2 - 1) R \left(\frac{r}{R} \right)^{2l-1} \sin^{2l-3} \theta \cos \theta, \quad (20)$$

yielding (18c). Now, inserting the above solution for $\delta^{(2)}v^\phi$ in eqs. (17a) and (17b) one obtains (19). Finally, through eq. (17c), $\delta^{(2)}v^r = 0$ implies $\delta^{(2)}v^\theta = 0$.

The above second-order solution, induced by first-order quantities, describes a drift of fluid elements along stellar latitudes. Note, however, that the homogeneous part of system (17), involving only second-order quantities, also admits a solution describing a drift along stellar latitudes,

$$\delta^{(2)}\hat{v}^r = 0, \quad (21a)$$

$$\delta^{(2)}\hat{v}^\theta = 0, \quad (21b)$$

$$\delta^{(2)}\hat{v}^\phi = \alpha^2 \Omega A r^{N-1} \sin^{N-1} \theta, \quad (21c)$$

with

$$\delta^{(2)}\hat{U} = \frac{2\alpha^2 \Omega^2 A}{N+1} r^{N+1} \sin^{N+1} \theta, \quad (22)$$

where A and N are some constants determined by initial data. Thus, the full second-order solution describing differential rotation has a piece induced by first-order quantities and another one which is a pure second-order effect. Since the differential rotation given by eq. (21c) is stratified on cylinders, it cannot cancel differential rotation induced by first-order quantities, which is *not* stratified on cylinders. Thus, differential rotation is an unavoidable feature of nonlinear r-modes.

The solution given by eqs. (18), (19), (21) and (22) is the nonlinear extension of the linear r-mode perturbation we were looking for.

The above solution also satisfies the perturbed continuity equation, which at lowest order in Ω is simply $\nabla_i(\rho \delta^{(2)}v^i) = 0$. Again, as in the linearized theory, the splitting of $\delta^{(2)}U$ into $\delta^{(2)}p$ and $\delta^{(2)}\Phi$ is done by using the perturbed Poisson equation (15) with appropriate boundary conditions for the gravitational potential $\delta^{(2)}\Phi$.

The second-order r-mode solution should also satisfy the zero-boundary condition for the pressure p at the surface of the star, i.e., the second-order Lagrangian change in pressure [29] should vanish at the surface of the star, $\Delta_\xi^{(2)}p = 0$.

The Lagrangian change in pressure is given by

$$\Delta_\xi p = \gamma p \frac{\Delta_\xi \rho}{\rho} + \frac{\gamma(\gamma-1)}{2} p \left(\frac{\Delta_\xi \rho}{\rho} \right)^2 + \dots, \quad (23)$$

where $\gamma = (\rho/p)(\partial p/\partial \rho)$. The Lagrangian change in density $\Delta_\xi \rho$ can be expressed in terms of the displacement vector using conservation of mass [25]:

$$\frac{\Delta_\xi \rho}{\rho} = -\nabla_k \xi^{(1)k} + \frac{1}{2} \left(\nabla_k \xi^{(1)k} \nabla_m \xi^{(1)m} + \nabla_k \xi^{(1)m} \nabla_m \xi^{(1)k} \right) - \nabla_k \xi^{(2)k} + \mathcal{O}(\xi^3), \quad (24)$$

where $\xi^{(2)i}$ are the contravariant components of the second-order Lagrangian displacement vector.

Using eq. (24) the second-order Lagrangian change in pressure $\Delta_\xi^{(2)}p$ is given by

$$\Delta_\xi^{(2)}p = \gamma p \left(\frac{1}{2} \nabla_k \xi^{(1)m} \nabla_m \xi^{(1)k} - \nabla_k \xi^{(2)k} \right) + \frac{\gamma^2}{2} p \nabla_k \xi^{(1)k} \nabla_m \xi^{(1)m}. \quad (25)$$

Since $\nabla_k \xi^{(1)k} = 0$, the boundary condition $\Delta_\xi^{(2)}p = 0$ is satisfied if

$$\nabla_k \xi^{(2)k} = \frac{1}{2} \nabla_k \xi^{(1)m} \nabla_m \xi^{(1)k}, \quad (26)$$

or, using eqs. (10), if

$$\nabla_k \xi^{(2)k} = \frac{1}{4} \alpha^2 C_l^2 l^2 (l+1)^2 \left(\frac{r}{R} \right)^{2l-2} [(l^2 - l + 1) \cos^2 \theta - l] \sin^{2l-4} \theta. \quad (27)$$

The contravariant components of the second-order Lagrangian displacement vector $\xi^{(2)i}$ are determined by the equations

$$\delta^{(2)}v^i = (\partial_t + \Omega \partial_\phi) \xi^{(2)i} - \xi^{(1)k} \nabla_k \delta^{(1)}v^i, \quad (28)$$

$$\delta^{(2)}\rho = -\rho \left(\nabla_k \xi^{(2)k} - \frac{1}{2} \nabla_k \xi^{(1)m} \nabla_m \xi^{(1)k} \right) - \left(\xi^{(2)k} \nabla_k \rho + \frac{1}{2} \xi^{(1)k} \xi^{(1)m} \nabla_k \nabla_m \rho \right), \quad (29)$$

where $\partial_t + \Omega\partial_\phi$ is the time derivative in the rotating frame, $\partial_{t'}$. In the above equations, terms quadratic in the first-order quantities $\xi^{(1)i}$ and $\delta^{(1)}v^i$ give rise to double “frequency” terms; since in this article we are concerned only with differential rotation, these double frequency terms will not be considered. Thus, at lowest order in Ω , eq. (28) yields the solution:

$$\xi^{(2)r} = C_1(r, \theta), \quad (30a)$$

$$\xi^{(2)\theta} = C_2(r, \theta), \quad (30b)$$

$$\xi^{(2)\phi} = D(r, \theta)t' + C_3(r, \theta), \quad (30c)$$

where

$$D(r, \theta) = \frac{1}{4}\alpha^2\Omega C_l^2 l^2(l+1)(2l-1) \left(\frac{r}{R}\right)^{2l-2} \sin^{2l-2}\theta + \alpha^2\Omega A r^{N-1} \sin^{N-1}\theta, \quad (31)$$

and C_i are some arbitrary functions of r and θ .

The second-order Lagrangian displacement vector $\xi^{(2)i}$, given by eqs. (30), is also a solution to eq. (29) at lowest order in Ω , if the functions C_i are chosen to be

$$C_1(r, \theta) = \frac{1}{16}\alpha^2 C_l^2 l^2(l+1)^2 R \left(\frac{r}{R}\right)^{2l-1} \sin^{2l-2}\theta (\sin^2\theta - 2), \quad (32a)$$

$$C_2(r, \theta) = \frac{1}{16}\alpha^2 C_l^2 l^2(l+1)^2 \left(\frac{r}{R}\right)^{2l-2} \sin^{2l-3}\theta \cos\theta (\sin^2\theta + 2l - 2), \quad (32b)$$

$$C_3(r, \theta) = 0. \quad (32c)$$

It is now straightforward to verify that $\xi^{(2)i}$, given by eqs. (30)–(32), satisfies the condition (27) everywhere in the star, including the surface. Thus, at second-order the solution satisfies the boundary condition $\Delta_\xi^{(2)}p = 0$ for any value of the constants A and N .

At second order, the Lagrangian change in vorticity is given by

$$q^i \equiv \Delta_\xi^{(2)}(\epsilon^{ijk}\nabla_j v_k) = \Delta_\xi^{(2)}\epsilon^{ijk}\nabla_j v_k + \epsilon^{ijk}\Delta_\xi^{(2)}\nabla_j v_k + \Delta_\xi^{(1)}\epsilon^{ijk}\Delta_\xi^{(1)}\nabla_j v_k, \quad (33)$$

where

$$\begin{aligned} \Delta_\xi^{(2)}\epsilon^{ijk} = & -\epsilon^{ijk}\nabla_m \xi^{(2)m} - \xi^{(1)n}\nabla_n (\epsilon^{ijk}\nabla_m \xi^{(1)m}) + \epsilon^{ijk}\nabla_m (\xi^{(1)n}\nabla_n \xi^{(1)m}) \\ & + \epsilon^{imn}\nabla_m \xi^{(1)j}\nabla_n \xi^{(1)k} + \epsilon^{mjn}\nabla_m \xi^{(1)i}\nabla_n \xi^{(1)k} + \epsilon^{mnk}\nabla_m \xi^{(1)i}\nabla_n \xi^{(1)j} \end{aligned} \quad (34)$$

and

$$\Delta_\xi^{(2)}\nabla_j v_k = \nabla_j \Delta_\xi^{(2)}v_k - v^m \nabla_j \nabla_k \xi_m^{(2)} - (\partial_t \xi^{(1)m} + v^n \nabla_n \xi^{(1)m}) \nabla_j \nabla_k \xi_m^{(1)}. \quad (35)$$

Using $\xi^{(1)i}$ and $\xi^{(2)i}$ given, respectively, by eqs. (10) and eqs. (30)–(32), and taking into account that the second-order Lagrangian change in velocity is given by [25]

$$\Delta_\xi^{(2)}v_i = \partial_t \xi^{(1)k} \nabla_i \xi_k^{(1)} + v^k \nabla_k \xi^{(1)m} \nabla_i \xi_m^{(1)} + \partial_t \xi_i^{(2)} + v^k (\nabla_i \xi_k^{(2)} + \nabla_k \xi_i^{(2)}), \quad (36)$$

one obtains for q^i the following expressions:

$$\begin{aligned} q^r = & \frac{1}{4}\alpha^2\Omega C_l^2 l^2(l+1) \left(\frac{r}{R}\right)^{2l-2} \sin^{2l-4}\theta \cos\theta [l(l+1)\sin^2\theta + 4(l-1)^2] \\ & + \alpha^2\Omega A(N+1)r^{N-1} \sin^{N-1}\theta \cos\theta, \end{aligned} \quad (37a)$$

$$\begin{aligned} q^\theta = & -\frac{1}{4}\alpha^2\Omega C_l^2 l^3(l+1)R^{-1} \left(\frac{r}{R}\right)^{2l-3} \sin^{2l-3}\theta [(l+1)\sin^2\theta + 4(l-1)] \\ & - \alpha^2\Omega A(N+1)r^{N-2} \sin^N\theta, \end{aligned} \quad (37b)$$

$$q^\phi = 0. \quad (37c)$$

As it can be seen from eqs. (37), the second-order Lagrangian change in vorticity is always different from zero for any values of the constants A and N . Thus, at second-order, r-modes do not preserve vorticity of fluid elements.

As it was mentioned in section II, the first-order canonical displacements $\xi^{(1)i}$ are precisely those that preserve vorticity. As a result, if the vorticity of a linear r-mode perturbation is initially small, then as the perturbation grows under gravitational radiation reaction the vorticity stays small. However, the situation changes when one takes into account second-order effects; due to differential rotation an initially small vorticity may increase as the perturbation grows.

IV. ANGULAR MOMENTUM AND ENERGY OF THE R-MODE PERTURBATION

The physical angular momentum of the second-order r-mode solution found in the previous section is, at lowest order in Ω , given by

$$\begin{aligned}\delta^{(2)}J &= \int \rho \delta^{(2)}v_\phi dV \\ &= \frac{1}{2}(l-1)(2l+1)\alpha^2\Omega R^{2-2l} \int_0^R \rho r^{2l+2} dr \\ &\quad + 2\pi\alpha^2\Omega A \int_0^R \rho r^{N+3} dr \int_0^\pi \sin^{N+2}\theta d\theta,\end{aligned}\tag{38}$$

where $\delta^{(2)}v_\phi$ is given by eqs. (18c) and (21c).

Since the physical angular momentum depends on the arbitrary constants A and N , one would like, at this point, to specify some physical conditions that fix the values of these constants, i.e., that pick out a member of the family of the second-order solution given by equations (18), (19), (21), and (22). Since in the linearized theory vorticity is conserved, it would be natural to impose the condition that the Lagrangian change in vorticity at every order should be zero. However, as we have seen, due to the presence of differential rotation, vorticity is not conserved already at second order.

The second-order physical angular momentum can be decomposed in two pieces [25]; one linear in the second-order Lagrangian change in velocity $\Delta_\xi^{(2)}v_i$ and another one, called the canonical angular momentum J_c , quadratic in the first-order Lagrangian displacement vector $\xi^{(1)i}$:

$$\delta^{(2)}J = \frac{1}{\Omega} \int \rho v^i \Delta_\xi^{(2)}v_i dV + J_c,\tag{39}$$

where

$$J_c = - \int \rho \partial_\phi \xi^{(1)i} \left(\partial_t \xi_i^{(1)} + v^k \nabla_k \xi_i^{(1)} \right) dV\tag{40}$$

and the second-order Lagrangian change in velocity is given by eq. (36).

For the Lagrangian displacement given by eqs. (10) and eqs. (30)–(32), the canonical angular momentum and the integral of the second-order Lagrangian change in velocity are computed to be, respectively,

$$J_c = -\frac{1}{4}l(l+1)\alpha^2\Omega R^{2-2l} \int_0^R \rho r^{2l+2} dr\tag{41}$$

and

$$\begin{aligned}\frac{1}{\Omega} \int \rho v^i \Delta_\xi^{(2)}v_i dV &= \frac{1}{4} (5l^2 - l - 2) \alpha^2 \Omega R^{2-2l} \int_0^R \rho r^{2l+2} dr \\ &\quad + 2\pi\alpha^2\Omega A \int_0^R \rho r^{N+3} dr \int_0^\pi \sin^{N+2}\theta d\theta.\end{aligned}\tag{42}$$

As it can be seen from the above equations, the second-order physical change in the angular momentum is not equal, in general, to the canonical angular momentum. Since the second-order physical change in the energy (in the inertial frame) $\delta^{(2)}E$ and the canonical energy E_c are related by the expression [25]

$$\delta^{(2)}E = \int \rho v^i \Delta_\xi^{(2)}v_i dV + E_c,\tag{43}$$

$\delta^{(2)}E$ and E_c are also, in general, not equal.

In recent years, the nonlinear behaviour of r-modes was investigated within a toy model of a spherical shell of rotating incompressible fluid [20]. It was shown that in a spherical shell r-modes carry zero physical angular momentum, $\delta^{(2)}J = 0$ [30], and positive physical energy, $\delta^{(2)}E > 0$, while both the canonical angular momentum and canonical energy are negative. Thus, for r-modes in a spherical shell one cannot equate physical and canonical quantities. Based on these results, it was conjectured in Ref. [20] that also in the case of a full star physical angular momentum and energy of r-modes are not equal to the corresponding canonical quantities. Our investigation, on r-modes at second order in α , confirms that for a full star, *in general*, $\delta^{(2)}J \neq J_c$ and $\delta^{(2)}E \neq E_c$. Note, however, that for specific choices of initial data [such that the two terms on the right hand side of eq. (42) cancel each other] physical and canonical quantities can be made equal.

V. DISCUSSION AND CONCLUSIONS

We have found a nonlinear extension of the linear r-mode perturbation, describing differential rotation of pure kinematic nature that produces large scale drifts along stellar latitudes. Differential rotation of fluid elements given by eqs. (18) is induced by first-order quantities, while differential rotation given by eqs. (21) is a pure second-order effect. The latter is stratified on cylinders and, therefore, cannot cancel differential rotation induced by first-order quantities, which is not stratified on cylinders. As already mentioned, our computation was carried out in a slow rotation approximation, $\Omega \ll \Omega_K$; in order to neglect higher-order terms arising from the expansion in powers of the star's angular velocity Ω , it was also assumed that $\alpha \gg (\Omega/\Omega_K)^2$ [see discussion before eq. (16)]. Recent results indicate that in rapidly rotating neutron stars (just born in a supernova or spun up by accretion in low-mass X-ray binaries) the r-mode instability may saturate at low values [10, 17], implying that α could be of the same order of magnitude as $(\Omega/\Omega_K)^2$. In that case, terms in $\delta^{(2)}v^i$ proportional to $\alpha\Omega^3$ (arising in an expansion in powers of the angular velocity of the star) are as important as terms proportional to $\alpha^2\Omega$ (arising in an expansion in powers of the mode amplitude) and, therefore, should be taken into account. However, even though the velocity field $\delta^{(2)}v^i$ and other second-order quantities derived in this paper are, strictly speaking, only valid in the regime $\Omega \ll \Omega_K$ and $\alpha \gg (\Omega/\Omega_K)^2$, they could be used to illustrate the influence of differential rotation of r-modes on the nonlinear evolution of rapidly rotating neutron stars.

Recently, an analytical expression for the azimuthal drift velocity of the $l = 2$ mode was derived from the linearized fluid equations by expanding the velocity of a fluid element located at a certain point in powers of α , averaging over a gyration, and retaining only the lowest-order nonvanishing term [17]. A comparison with our results shows the expression obtained in Ref. [17] is qualitatively correct but not exact to $\mathcal{O}(\alpha^2)$. This is to be expected since the procedure used there does not consider nonlinear effects in the fluid equations.

In the linearized theory, r-mode perturbation preserves vorticity of fluid elements, with the consequence that the vorticity will not grow as the perturbation grows under gravitational radiation reaction. However, the situation changes when one takes into account second-order effects; due to differential rotation of fluid elements, producing large scale azimuthal drifts, the second-order Lagrangian change in vorticity is always different from zero and may increase as the perturbation grows under gravitational radiation reaction.

It was also explicitly shown that for r-modes physical and canonical quantities cannot be equated. Canonical angular momentum (or energy) is not the full angular momentum (or energy) at second order; one should also include a part linear in the second-order Lagrangian change in velocity, which, as pointed out in Ref. [25], is related to conservation of circulation in the fluid. Since, as shown above, at second-order r-modes do not conserve vorticity, it follows that, in general, the physical and canonical angular momentum (or energy) do not coincide. However, specific choices of A and N can be made such that the integral in the right-hand side of eqs. (39) and (43) vanishes. Such a case (for $l = 2$) was studied in Ref. [9] within a phenomenological model for the evolution of r-modes. However, it is not clear which physical condition would force the constants A and N to take such particular values; thus, it seems more appropriate to study the evolution of r-modes for the case of arbitrary values of A and N .

In recent years, numerical simulations have been used to study the nonlinear evolution of r-modes, both for relativistic and Newtonian stars. Initial data for this evolution was generated by adding a linear perturbation $\delta^{(1)}v^i$ to an equilibrium stellar model. An analytical expression for $\delta^{(1)}v^i$ is known only for small-amplitude perturbations of slowly-rotating Newtonian stars. As shown in this paper, the linear r-mode perturbations given by eq. (4) induce, at second order in the mode amplitude, a drift of fluid elements along stellar latitudes, which cannot be avoided by special choices of the constants A and N . Thus, initial data for numerical non-linear evolution of r-modes should also include a piece describing differential rotation.

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- [28] Throughout this article, we use the definitions and notation of Ref. [25]. In particular, the first-order Lagrangian change in a quantity q is defined as $\Delta_{\xi}^{(1)} q = \delta^{(1)} q + \mathcal{L}_{\xi^{(1)}} q$, where $\xi^{(1)}$ is the first-order Lagrangian displacement vector which connects fluid elements in the equilibrium with corresponding ones in the perturbed configuration; the action of the Lie derivative $\mathcal{L}_{\xi^{(1)}}$ on a tensor $T^{a_1 \dots a_k}_{b_1 \dots b_l}$ is given by $\mathcal{L}_{\xi^{(1)}} T^{a_1 \dots a_k}_{b_1 \dots b_l} = \xi^{(1)c} \nabla_c T^{a_1 \dots a_k}_{b_1 \dots b_l} - \sum_{i=1}^k T^{a_1 \dots c \dots a_k}_{b_1 \dots b_l} \nabla_c \xi^{(1)a_i} + \sum_{j=1}^l T^{a_1 \dots a_k}_{b_1 \dots c \dots b_l} \nabla_{b_j} \xi^{(1)c}$. Note that the above definition of Lagrangian change agrees with the one commonly used in some literature of stellar pulsation, $\Delta_{\xi}^{(1)} q = \delta_{\xi}^{(1)} q + \xi^{(1)a} \nabla_a q$, only for scalars.
- [29] In this section, the second-order Lagrangian change in a quantity q is derived using the formalism developed in Appendix B of Ref. [25], taking into account that it contains, besides terms quadratic in the first-order Lagrangian displacement $\xi^{(1)i}$, terms linear in the second-order Lagrangian displacement $\xi^{(2)i}$ [27]. For scalars f and contravariant vector fields v^i the relation between the second-order Eulerian and Lagrangian changes is given, respectively, by $\Delta_{\xi}^{(2)} f = \delta^{(2)} f + \xi^{(2)a} \nabla_a f + 1/2 \xi^{(1)a} \xi^{(1)b} \nabla_a \nabla_b f + \xi^{(1)a} \nabla_a \delta^{(1)} f$ and $\Delta_{\xi}^{(2)} v^i = \delta^{(2)} v^i + \xi^{(2)a} \nabla_a v^i - v^a \nabla_a \xi^{(2)i} - \nabla_a \xi^{(1)i} (\xi^{(1)b} \nabla_b v^a - v^b \nabla_b \xi^{(1)a}) + \xi^{(1)a} \nabla_a \delta^{(1)} v^i - \delta^{(1)} v^a \nabla_a \xi^{(1)i}$.
- [30] In a spherical shell, the r-mode solution $v^{\theta} \propto \alpha \Omega \sin^{l-1} \theta \sin(l\phi + \omega t)$ and $v^{\phi} \propto \alpha \Omega \sin^{l-2} \theta \cos \theta \cos(l\phi + \omega t)$, which is exact for arbitrary amplitude of the mode, gives a zero contribution to the physical angular momentum of the shell. In a full star, the above solution (with $v^r = 0$) is just a first order approximation and one has to take into account higher order terms, the axisymmetric part of which does contribute to the physical angular momentum of the star.